Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Friday 7 June 2024

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/3C**

Further Mathematics

Advanced

PAPER 3C: Further Mechanics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Unless otherwise indicated, whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$ and give your answer to either 2 significant figures or 3 significant figures.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

P 7 5 3 2 2 A 0 1 2 4

Pearson

1. [*In this question,* **i** *and* **j** *are horizontal perpendicular unit vectors.*]

A particle A has mass 3 kg and a particle B has mass 2 kg.

The particles are moving on a smooth horizontal plane when they collide directly.

Immediately **before** the collision, the velocity of A is $(3\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$ and the velocity of B is $(-6\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$

Immediately **after** the collision the velocity of A is $\left(-2\mathbf{i} + \frac{2}{3}\mathbf{j}\right)\text{m s}^{-1}$

(a) Find the total kinetic energy of the two particles **before** the collision. **(3)**

(b) Find, in terms of **i** and **j**, the impulse exerted on A by B in the collision. **(3)**

(c) Find, in terms of **i** and **j**, the velocity of B immediately **after** the collision. **(3)**

first of all, let's illustrate the above OBLIQUE COLLISION diagrammatically - label direction of motion, respective velocities, etc.

BEFORE:

AFTER:

(a) first we're asked to find the total K.E of the particle before the collision; subbing into our formula for K.E

formula: $K.E = \frac{1}{2} m v^2$ (m: MASS in kg; v: scalar VELOCITY in ms^{-1})

before we sub in, PYTHAGORISE the velocities BEFORE for A and B so we can sub in the scalar velocities, not the vectors

$|v_A| = \sqrt{(3)^2 + (-1)^2}$ $\quad$ $|v_B| = \sqrt{(-6)^2 + (2)^2}$

$= \sqrt{10}$ $\quad$ $= \sqrt{40}$

subbing this into our formula:

$K.E = \frac{1}{2}(3)(\sqrt{10})^2 + \frac{1}{2}(2)(\sqrt{40})^2$

expand brackets

$= 3/2(10) + (40) = \boxed{55\text{ J}}$

units for K.E is Joules, J

(b) now to find the Impulse exerted on A by B, let's sub in the information given on A into our Impulse-momentum formula (VECTOR version)

↳ NOTE: could in theory also have worked out the Impulse exerted on B by A (due to Newton's Third Law which states that every action has an equal and opposite reaction), but the question gives us more information on A

formula: $\underline{I} = m(\underline{v} - \underline{u})$

subbing into above:

$$\underline{I} = 3\left(\begin{pmatrix} -2 \\ 2/3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right)$$

expanding

$$\underline{I} = 3\begin{pmatrix} -5 \\ 5/3 \end{pmatrix} = \boxed{\begin{pmatrix} -15 \\ 5 \end{pmatrix} \text{Ns}}$$

units for Impulse are Newton seconds, Ns

(c) let $\underline{v}_B = \begin{pmatrix} a \\ b \end{pmatrix}$ - two ways to find this vector velocity

WAY 1: subbing into vector PCLM

formula: $m_A\underline{u}_A + m_B\underline{u}_B = m_A\underline{v}_A + m_B\underline{v}_B$

(means total momentum before the collision equals total momentum after)

$$3\begin{pmatrix} 3 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -6 \\ 2 \end{pmatrix} = 3\begin{pmatrix} -2 \\ 2/3 \end{pmatrix} + 2\begin{pmatrix} a \\ b \end{pmatrix}$$

expand above

$$\begin{pmatrix} 9 \\ -3 \end{pmatrix} + \begin{pmatrix} -12 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix} + \begin{pmatrix} 2a \\ 2b \end{pmatrix}$$

collect vectors

$$\begin{pmatrix} 2a \\ 2b \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\div 2$ $\div 2$

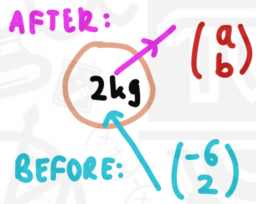
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} \text{ms}^{-1}$$

or $\begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix} \text{ms}^{-1}$

$$\boxed{\therefore \underline{v}_B = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} \text{ms}^{-1} \quad \text{or} \quad \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix} \text{ms}^{-1}}$$

WAY 2: vector impulse-momentum (on B by A)

...focusing just on the motion of B:



subbing into our vector formula for Impulse-momentum

formula: $\underline{I} = m(\underline{v} - \underline{u})$

↳ here we can exploit Newton's 3rd Law - means every action has an equal and opposite reaction

↳ ∴ -ve

$\Rightarrow$ $\underline{I}$ on B by A $= -(\underline{I}$ on A by B$)$

and know from (b) that $\underline{I} = -\begin{pmatrix} -15 \\ 5 \end{pmatrix} = \begin{pmatrix} 15 \\ -5 \end{pmatrix}$

hence subbing into our formula:

$$\begin{pmatrix} 15 \\ -5 \end{pmatrix} = 2\left(\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} -6 \\ 2 \end{pmatrix}\right)$$

expand above

$$\begin{pmatrix} 15 \\ -5 \end{pmatrix} = 2\begin{pmatrix} a+6 \\ b-2 \end{pmatrix}$$

$$\begin{pmatrix} 15 \\ -5 \end{pmatrix} = \begin{pmatrix} 2a+12 \\ 2b-4 \end{pmatrix}$$

equating $\underline{i}$ components

$\underline{i}$: $15 = 2a + 12$

$\Rightarrow 2a = 3$

$\div 2$ $\div 2$

$a = 3/2$

equating $\underline{j}$ components

$\underline{j}$: $-5 = 2b - 4 \Rightarrow 2b = -1$

Question 1 continued

$\Rightarrow b = -\frac{1}{2}$

$\therefore$ subbing into our initial expression for v_B

$$\boxed{v_B = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} \text{ or } \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix} \text{ ms}^{-1}}$$

(Total for Question 1 is 9 marks)

2. A rough plane is inclined to the horizontal at an angle θ, where $\tan\theta = \frac{3}{4}$

A particle P of mass m is at rest at a point on the plane.

The particle is projected **up** the plane with speed $\sqrt{2ag}$

The particle moves up a line of greatest slope of the plane and comes to instantaneous rest after moving a distance d.

The coefficient of friction between P and the plane is $\frac{1}{7}$

(a) Show that the magnitude of the frictional force acting on P as it moves up the plane is $\frac{4mg}{35}$

(3)

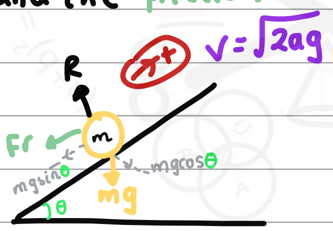
Air resistance is assumed to be negligible.

Using the work-energy principle,

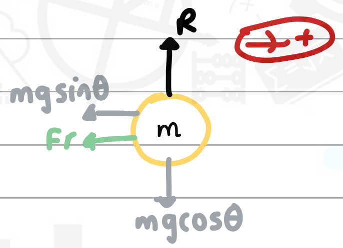
(b) find d in terms of a.

(4)

(a) let's illustrate the above diagrammatically - label the speed, the distance, 'd' and the friction



turn the axis



the question asks us to find Fr, where:

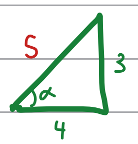
formula: $Fr = \mu R$

coefficient of friction: $\mu = 1/7$

reaction force: have to resolve vertically:

$R(\uparrow)$: $R = mg\cos\theta$

but given that $\tan\theta = 3/4$ - hence constructing the appropriate trig triangle, using the 3, 4, 5 Pythag. triple



$\Rightarrow \cos\alpha = A/H = 4/5$

$\sin\alpha = O/H = 3/5$

$\therefore R = mg(4/5)$

$= 4/5\, mg$

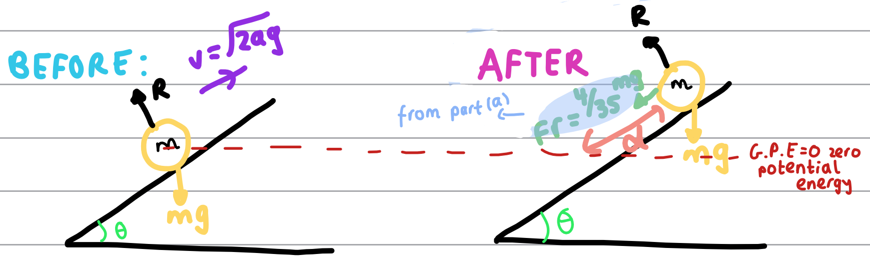
$\therefore Fr = 1/7 \times 4/5\, mg = \boxed{4/35\, mg}$

P 7 5 3 2 2 A 0 4 2 4

DO NOT WRITE IN THIS AREA

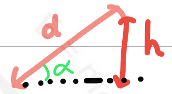
Question 2 continued

(b) to use the work-energy principle, let's draw out two diagrams: one for before the particle travels the distance 'd' and one for after - label the appropriate energies



• K.E

• G.P.E - need perp.distance 'h'



$\Rightarrow h = d\sin\alpha$

but from (a) know

$\sin\alpha = 3/5$

$\therefore h = 3/5\, d$

• work done by $Fr = 4/35\, mg$

now sub all into work-energy principle (includes dissipative forces)

$\underset{n/a}{w.d\ in} + \underset{\text{initial kinetic}}{K.E_i} + \underset{\text{initial gravitational potential}}{G.P.E_i} + \underset{\text{initial elastic potential}}{E.P.E_i} = \underset{\text{final kinetic energy}}{K.E_f} + \underset{\text{final gravitational potential}}{G.P.E_f} + \underset{\text{final elastic potential}}{E.P.E_f} + \text{w.d against friction}$

formula: $\frac{1}{2}mu^2 + mgh_1 + \frac{\lambda x^2}{2l} = \frac{1}{2}mv^2 + mgh_2 + \frac{\lambda x^2}{2l} + Fr \times d$

sub into above

need to find this!

$\frac{1}{2}\cancel{m}(\sqrt{2ag})^2 + 0 + 0 = 0 + \cancel{m}g(3/5\, d) + 4/35\, \cancel{m}gd$

cancel m's and expand brackets

$a\cancel{g} = 3/5\, \cancel{g}d + 4/35\, \cancel{g}d$

cancel g's and collect like terms

$a = 5/7\, d$

$\div 5/7 \qquad \div 5/7$

$\Rightarrow \boxed{d = 7/5\, a}$

(Total for Question 2 is 7 marks)

3. A car of mass 1000 kg moves in a straight line along a horizontal road at a constant speed of $72\,\text{km h}^{-1}$

- The resistance to the motion of the car is modelled as a constant force of magnitude 900 N

The engine of the car is working at a constant rate of P kW.

Using the model,

(a) find the value of P.

(3)

The car now travels in a straight line up a road which is inclined to the horizontal at an angle α, where $\sin\alpha = \frac{2}{49}$

- In a refined model, the resistance to the motion of the car from non-gravitational forces is now modelled as a force of magnitude $20v$ newtons, where $v\,\text{m s}^{-1}$ is the speed of the car

At the instant when the engine of the car is working at a constant rate of 30 kW and the car is moving up the road at $10\,\text{m s}^{-1}$, the acceleration of the car is $a\,\text{m s}^{-2}$

Using the refined model,

(b) find the value of a.

(4)

Later on, when the engine of the car is again working at a constant rate of 30 kW, the car is moving up the road at a constant speed $U\,\text{m s}^{-1}$

Using the refined model,

(c) find the value of U.

(5)

(a) let's illustrate the above diagrammatically: make sure to label the weight, the resistance to motion, and the Power rearranged

↳ formula: $P = Fv$ (P: power in watts; F: force in Newtons; v: velocity in ms^{-1})

$F = \frac{P}{v}$

...where:

$P\text{kW} \xrightarrow[\text{watts} \times 1000]{\text{convert to}} P000\text{W}$

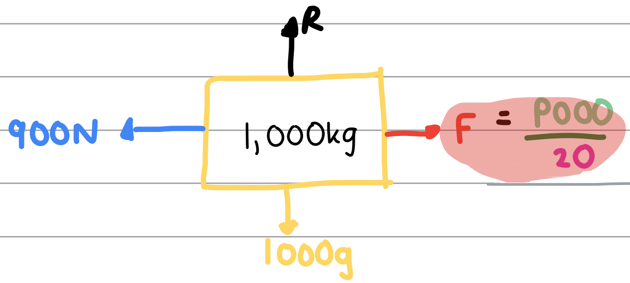
and $v = 72\text{kmh}^{-1}$

let's convert 72kmh^{-1} to ms^{-1}

$72\text{kmh}^{-1} \xrightarrow{\times 1000} 72{,}000\text{mh}^{-1} \xrightarrow{\div 3{,}600} 20\text{ms}^{-1}$

P 7 5 3 2 2 A 0 6 2 4

Question 3 continued CAR:



NOTE: could've done this as a separate line of working but it's more efficient in the exam to write in the FORCE from the POWER rearranged

now we use the fact that because the car is moving at a constant speed, this means that it's in non-stationary equilibrium, so forces left = forces right

$$\Rightarrow \frac{P000}{20} = 900$$

$\times 20$ $\times 20$

$$P000 = 18{,}000$$

$\div 1000$ $\div 1000$

$$\boxed{\therefore\ P = 18\,kW}$$

(b) now the car is moving UP the INCLINED PLANE – illustrating this diagrammatically – label the weight resolved, resistance to motion, power REARRANGED

$\underbrace{\text{resistance to motion}}$: 20v from question

but this is variable resistance

$\therefore$ Sub in $v = 10ms^{-1}$

$\therefore 20(10)$

$= 200N$

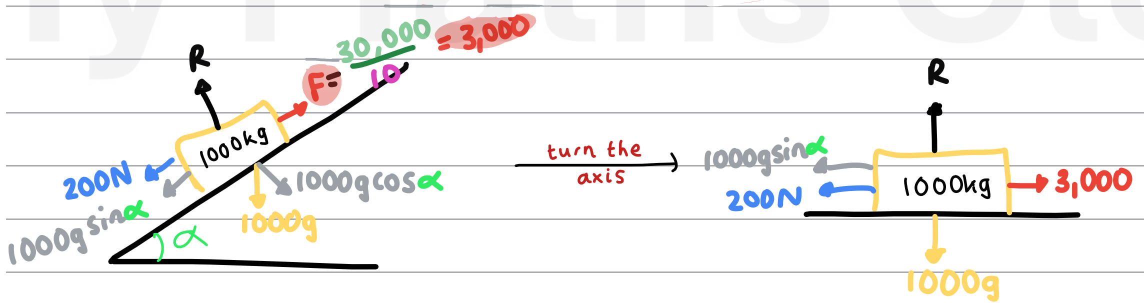
$\underbrace{\text{power REARRANGED}}$: formula: $P = Fv$

$F = \frac{P}{v}$

...where:

$30kW \xrightarrow[\times 1000]{\text{convert to watts}} 30{,}000W$

and $v = 10ms^{-1}$



...to find the acceleration let's resolve horizontally – subbing into Newton's Second Law:

$$\Sigma F_x = 3{,}000 - 200 - 1000g\sin\alpha = 1000a$$

$\hookrightarrow \sin\alpha = 2/49$ } given in question

P 7 5 3 2 2 A 0 7 2 4

Question 3 continued

collect like terms:

$$2800 - \frac{2000}{49}g = 1000a$$

$$\therefore 2400 = 1000a$$

$\div 1000$ $\quad$ $\div 1000$

$$\Rightarrow \boxed{a = 2.4\,ms^{-2}}$$

(c) see how in the refined model, the only thing that changes is v becomes u → this hence affects the:

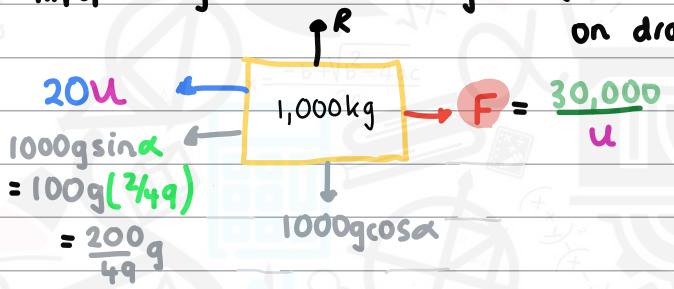
- $F = \frac{P}{u}$

$30kW \xrightarrow[\times 1000]{\text{convert to Watts}} 30{,}000W$

and $v = u$

- and resistance $= 20u$

...populating onto our diagram (NOTE: in exam, don't waste time on drawing a second diagram - just alter the force and resistance components of the diagram from part (b))



now use the fact that the car is moving with constant speed ∴ suggests it's in non-stationary equilibrium, so forces left = forces right

$$R(\leftarrow): \quad \frac{30{,}000}{u} = 20u + \frac{2000}{49}g$$

$\times u$ $\quad$ $\times u$

$$20u^2 + \frac{2000}{49}gu - 30{,}000 = 0$$

$$\Rightarrow 20u^2 + 400u - 30{,}000 = 0$$

$\div 20$ $\quad$ $\div 20$

$$u^2 + 20u - 1{,}500 = 0$$

solving this quadratic equtn
↳ it's easily factorisable

$$(u-30)(u+50) = 0$$

$$\Rightarrow \boxed{u = 30\,ms^{-1}} \text{ or } -5\cancel{0ms}^{-1} \text{ reject -ve}$$

Question 3 continued

(Total for Question 3 is 12 marks)

4. A particle A of mass $2m$ is moving in a straight line with speed $3u$ on a smooth horizontal plane. Particle A collides directly with a particle B of mass m which is at rest on the plane.

The coefficient of restitution between A and B is e, where $e > 0$

(a) Show that the speed of B immediately after the collision is $2u(1 + e)$.

(6)

After the collision, B hits a smooth fixed vertical wall which is perpendicular to the direction of motion of B.

(b) Show that there will be a second collision between A and B.

(3)

The coefficient of restitution between B and the wall is $\frac{1}{2}$

Find, **in simplified form**, in terms of m, u and e,

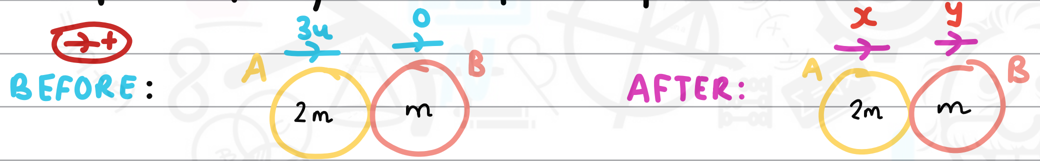
(c) the magnitude of the impulse received by B in its collision with the wall,

(3)

(d) the loss in kinetic energy of B due to its collision with the wall.

(3)

let's illustrate this elastic collision in 1D diagrammatically – make sure to label the respective speeds, direction of motion, etc.



following the normal procedure for this type of collision:

... first using PCLM - this means that the total momentum before equals the total momentum after:

formula: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$

sub into the above

$2\cancel{m}(3u) = 2\cancel{m}x + \cancel{m}y$

cancel m's and expand brackets

$\Rightarrow 2x + y = 6u$ - ①

next, because both the speeds after are unknown, let's use NEL (Impact Law) rearranged

formula: $e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$

subbing into above

Question 4 continued

$$e = \frac{y-(x)}{3u-0} = \frac{y-x}{3u}$$

$$\therefore y - x = 3eu \quad -②$$

and because we're asked for $v_B = y$, need to solve ① and ② simultaneously → elim. 'x'

①+2×②

$$2x + y = 6u$$
$$+\ -2x + 2y = 6eu$$
$$3y = 6u + 6eu$$

factorise 6u out

$$3y = 6u(1+e)$$
$$\div 3 \qquad \div 3$$
$$y = \frac{6u}{3}(1+e)$$
$$\Rightarrow y = 2u(1+e)$$
$$\boxed{\therefore v_Q = 2u(1+e)}$$

(b) the fact that particle B may potentially collide with A a second time suggests we should find the value of v_A or x

↳ hence getting the ① and ② from (a) and elim. 'y'

①−②

$$2x + y = 6u$$
$$-\ -x + y = 3eu$$
$$3x = 6u - 3eu$$

factorise 3u out

$$3x = 3u(2-e)$$
$$\div 3 \qquad \div 3$$
$$\Rightarrow x = u(2-e)$$

and because $0 \le e \le 1$ (our source of inequality)

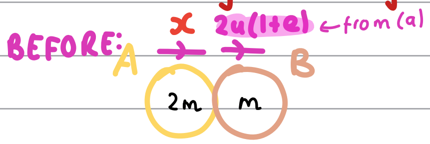
$$\Rightarrow x > 0$$

↳ hence A does NOT change direction after the collision, but B does (as it rebounds off the wall) hence there must be a second collision between A and B

Question 4 continued

(c) now that we're given the value of the coefficient of restitution this means we can find w_B i.e the velocity AFTER B's collision with the wall to compare with v_A or x

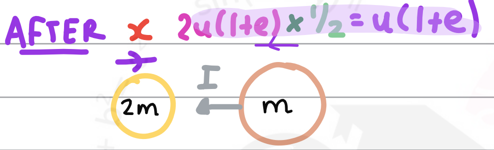
↳ illustrating this diagrammatically:



$e = 1/2$

find velocity AFTER for B - this is when NEL rearranged applies

$\therefore$ multiply v_B by $e = 1/2$



(←+)

subbing into our formula for Impulse-momentum

formula: $I = m(v - u)$

sub into above

$I = m(-u(1+e) - 2u(1+e))$

expanding above

$\Rightarrow I = m(-u - eu - 2u - 2eu)$

collect like terms

$I = m(-3u - 3eu)$

factorise $-3u$ out

$I = -3mu(1+e)$ Ns

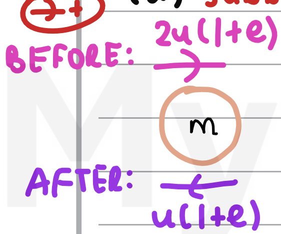
but asked for magnitude

$\boxed{\therefore I = 3mu(1+e)}$ Ns

units for Impulse: Newton seconds, Ns

(d) subbing into the formula for K.E lost

(→+)



formula: $\frac{1}{2}m(u^2 - v^2)$

subbing into above:

K.E lost $= \frac{1}{2}m((2u(1+e))^2 - (-u(1+e))^2)$

expanding brackets

$= \frac{1}{2}m(4u^2(1+e)^2 - u^2(1+e)^2)$

collect like expressions

$= \frac{1}{2}m(3u^2(1+e)^2)$

units for energy is Joules, J

$\boxed{= \frac{3}{2}mu^2(1+e)^2 \text{ J lost}}$

Question 4 continued

(Total for Question 4 is 15 marks)

5. A light elastic string has natural length $2a$ and modulus of elasticity $2mg$.
One end of the string is attached to a fixed point A on a horizontal ceiling.
The other end is attached to a particle P of mass m.

The particle P hangs in equilibrium at the point E, where $AE = 3a$.

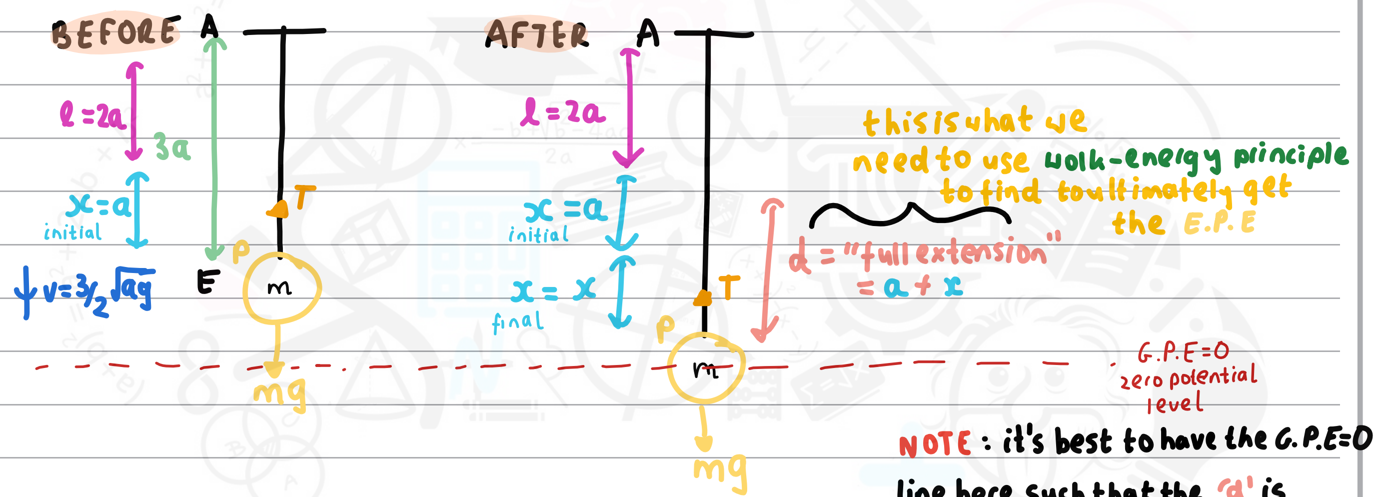
The particle P is then projected vertically downwards from E with speed $\frac{3}{2}\sqrt{ag}$

Air resistance is assumed to be negligible.

Find the elastic energy stored in the string, when P first comes to instantaneous rest.
Give your answer in the form $kmga$, where k is a constant to be found.

(7)

the most important thing to consider when approaching elastic strings and springs questions is to draw a detailed diagram - here let's do one for BEFORE the particle is projected and one for AFTER - label the appropriate energies



NOTE: it's best to have the G.P.E=0 line here such that the 'a' is in terms of the extension rather than the extension being in terms of 'd'

- E.P.E
- K.E
- G.P.E

- E.P.E

now sub all into work-energy principle (includes dissipative forces)

$$\underset{n/a}{\text{w.d in}} + \underset{\text{initial kinetic}}{K.E_i} + \underset{\text{initial gravitational potential}}{G.P.E_i} + \underset{\text{initial elastic potential}}{E.P.E_i} = \underset{\text{final kinetic energy}}{K.E_f} + \underset{\text{final gravitational potential}}{G.P.E_f} + \underset{\text{final elastic potential}}{E.P.E_f} + \text{w.d against friction}$$

subbing into above formula: $\frac{1}{2}mu^2 + mgh_1 + \frac{\lambda x^2}{2l} = \frac{1}{2}mv^2 + mgh_2 + \frac{\lambda x^2}{2l} + Fr \times d$

($x_{initial}$, x_{final})

$$\frac{1}{2}m\left(\frac{3}{2}\sqrt{ag}\right)^2 + mgx + \frac{2mg(a)^2}{2(2a)} = \frac{2mg(a+x)^2}{2(2a)}$$

expand brackets

what we're trying to find!

$$\frac{9}{8}\cancel{m}ga + \cancel{m}g(x) + \frac{\cancel{m}ga}{2} = \frac{2\cancel{m}g(a+x)^2}{4a}$$

P 7 5 3 2 2 A 0 1 4 2 4

Question 5 continued

cancel mg's

$$\frac{9}{8}a + x + \frac{a}{2} = \frac{\cancel{2}(a+x)^2}{\cancel{4}a}$$

Solve above for 'x'

$$\frac{9}{8}a + x = \frac{(a+x)^2}{2a} - \frac{a}{2}$$

expand the double bracket

$$\frac{9}{8}a + x = \frac{a^2+2ax+x^2}{2a} - \frac{a}{2}$$

get common denominator on RHS

$$\frac{9}{8}a + x = \frac{\cancel{a^2}+2ax+x^2-\cancel{a^2}}{2a}$$

$\times 2a$ $\times 2a$

$$\frac{9}{4}a^2 + \cancel{2ax} = \cancel{2ax} + x^2$$

$$\Rightarrow x^2 = \frac{9}{4}a^2$$

Square root both sides

$$x = 3/_2 a$$

$$\therefore d = a + 3/_2 a$$

$$= 5/_2 a$$

subbing this into our formula for E.P.E

formula: $E.P.E = \frac{\lambda x^2}{2l}$ mod. of elasticity, extension, natural length

$$= \frac{\cancel{2}mg(5/_2 a)^2}{\cancel{2}(2a)}$$

Simplify down

$$= \frac{\frac{25}{4}mga}{2} = \boxed{\frac{25}{8}mga \text{ J}}$$

units for energy are Joules, J

(Total for Question 5 is 7 marks)

6. [*In this question,* **i** *and* **j** *are horizontal perpendicular unit vectors.*]

A particle P is moving with velocity $(4\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$ on a smooth horizontal plane. The particle collides with a smooth vertical wall and rebounds with velocity $(\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$

The coefficient of restitution between P and the wall is e.

(a) Find the value of e.

(6)

After the collision, P goes on to hit a second smooth vertical wall, which is parallel to **i**.

The coefficient of restitution between P and this second wall is $\frac{1}{3}$

The angle through which the direction of motion of P has been deflected by its collision with this second wall is $\alpha°$.

(b) Find the value of α, giving your answer to the nearest whole number.

(4)

(a) illustrating this oblique collision with a fixed surface diagrammatically - label the velocity components, direction of motion, etc.

the key here is that we are dealing with an unknown vector wall - hence need to find the unknowns by subbing into our two formulae for collisions with vector walls

formula: $u \cdot w = v \cdot w$
vector wall

subbing into above

$\binom{4}{-1} \cdot \binom{a}{b} = \binom{1}{3} \cdot \binom{a}{b}$

evaluating the scalar product

$4a - b = a + 3b$

collect like terms

$3a = 4b$

$\div 4 \quad \div 4$

$b = \frac{3}{4}a$

... can use this info in 2 ways to get the vector wall:

WAY 1: as a vector

if $b = \frac{3}{4}a$, then

$w = \binom{1}{3/4}$

$\times 4 \quad \times 4$

$= \binom{4}{3}$

WAY 2: as a linear equation

if $b = \frac{3}{4}a$, this is the same as

$y = \frac{3}{4}x$

$\therefore m = \frac{3}{4}$ ← y ← x

$\therefore$ as a vector $= \binom{x}{y} = \binom{4}{3}$

P 7 5 3 2 2 A 0 1 6 2 4

DO NOT WRITE IN THIS AREA

Question 6 continued

now we've got $w = \begin{pmatrix}4\\3\end{pmatrix}$, this means that the Impulse is the vector perpendicular to w - 3 ways to find this:

WAY 1: dot product

formula: $a \cdot b = 0$

$\begin{pmatrix}4\\3\end{pmatrix} \cdot I = 0$

$\Rightarrow I = \begin{pmatrix}-3\\4\end{pmatrix}$

or $\begin{pmatrix}3\\-4\end{pmatrix}$

WAY 2: as linear equtns

$\begin{pmatrix}4\\3\end{pmatrix} \therefore m_1 = 3/4$

$\therefore y = 3/4 x$

formula: $m_1 \times m_2 = -1$

$m_2 =$ negative reciprocal of $m_1 \therefore -4/3$

$\Rightarrow \begin{pmatrix}1\\-4/3\end{pmatrix}$ $\times -3$

$\therefore \begin{pmatrix}-3\\4\end{pmatrix}$ as a vector

or $\times 3$

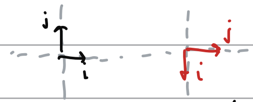
$\begin{pmatrix}3\\-4\end{pmatrix}$

$\therefore I = \begin{pmatrix}-3\\4\end{pmatrix}$ or $\begin{pmatrix}3\\-4\end{pmatrix}$

WAY 3: 90° rotation

treating the 'perpendicular' vector as a 90° rotation of $\begin{pmatrix}4\\3\end{pmatrix}$

... clockwise:

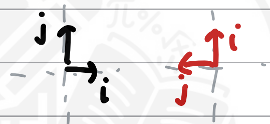


$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$

$\therefore$ as a linear transformation

$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}4\\-3\end{pmatrix}$

... anticlockwise:



$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

$\therefore$ as a linear transformation

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}-3\\4\end{pmatrix}$

$\therefore I = \begin{pmatrix}4\\-3\end{pmatrix}$ or $\begin{pmatrix}-3\\4\end{pmatrix}$

proceed with ANY of the two, eg. $I = \begin{pmatrix}-3\\4\end{pmatrix}$

now can sub into our second formula:

formula: $-eu \cdot I = v \cdot I$

subbing into above

$-e\begin{pmatrix}4\\-1\end{pmatrix} \cdot \begin{pmatrix}-3\\4\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \cdot \begin{pmatrix}-3\\4\end{pmatrix}$

evaluating the scalar product

$-e(-16) = 9$

expanding and solving for e:

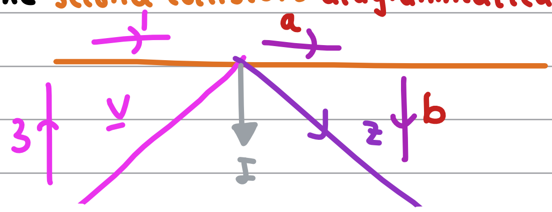
$16e = 9$

$\div 16 \quad \div 16$

$\boxed{e = 9/16}$

Question 6 continued

(b) illustrating the second collision diagrammatically - now with a NON-VECTOR WALL



$e = 1/3$

the fact that we need to find the angle of deflection suggests we need to find the parallel and perp. components of the z_p

...two ways to do this:

WAY 1: using the standard non-vector wall formulae

...parallel:

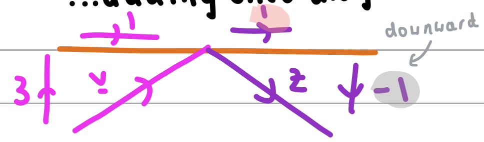
no change $\therefore = 1$

...perpendicular:

NEL rearranged applies (due to the Impulse acting perpendicular to the wall

$\therefore 3 \times 1/3 = 1$

...adding onto diagram:



$\Rightarrow z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

WAY 2: using vector formulae:

now $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\therefore I =$ perpendicular $= \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

and let $z = \begin{pmatrix} a \\ b \end{pmatrix}$

subbing above into formula:

formula: $u \cdot w = v \cdot w$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

evaluate the scalar products

$\Rightarrow \boxed{1 = a}$

...now, next formula:

formula: $-e\, u \cdot I = v \cdot I$

subbing into above:

$-1/3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

evaluate the scalar products

$-1/3(-3) = -b$

$\Rightarrow 1 = -b$

$\boxed{\therefore b = -1}$

$\therefore z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

now need to find the angle of deflection between

$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

↳ two ways to do this:

WAY 1: scalar dot product formula

formula: $\cos\alpha = \dfrac{a \cdot b}{|a||b|}$ ← scalar dot product ← product of magnitudes

WAY 2: vector triangles and trig

see α is the sum of the angle in the pink triangle - call it 'θ'

P 7 5 3 2 2 A 0 1 8 2 4

Question 6 continued

subbing into above:

$$\cos\alpha = \frac{\begin{pmatrix}1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}}{\sqrt{(1)^2+(3)^2}\sqrt{(1)^2+(-1)^2}}$$

evaluate scalar product and square roots

$$\cos\alpha = \frac{-2}{\sqrt{10}\sqrt{2}} = \frac{-2}{\sqrt{20}}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{-2}{\sqrt{20}}\right)$$

$$= 116.565\ldots$$

$$\boxed{= 117^\circ (3\ s.f)}$$

and angle in the purple triangle – call it 'θ'

$$\Rightarrow \alpha = \tan^{-1}(3/1) + \tan^{-1}(1/1)$$

$$= 71.56501\ldots^\circ + 45^\circ$$

$$= \boxed{117^\circ} (3\ s.f)$$

(Total for Question 6 is 10 marks)

P 7 5 3 2 2 A 0 1 9 2 4

7.

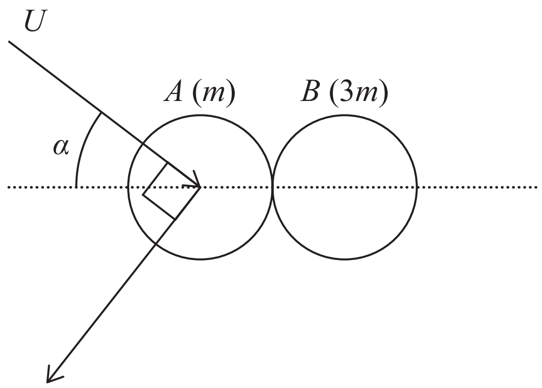


Figure 1

A smooth uniform sphere A of mass m is moving with speed U on a smooth horizontal plane. The sphere A collides obliquely with a smooth uniform sphere B of mass $3m$ which is at rest on the plane. The two spheres have the same radius.

Immediately before the collision, the direction of motion of A makes an angle α, where $0° < \alpha < 90°$, with the line joining the centres of the spheres.

Immediately after the collision, the direction of motion of A is **perpendicular** to its original direction, as shown in Figure 1.

The coefficient of restitution between the spheres is e.

(a) Show that the speed of B immediately after the collision is

$$\frac{1}{4}(1+e)U\cos\alpha$$

(6)

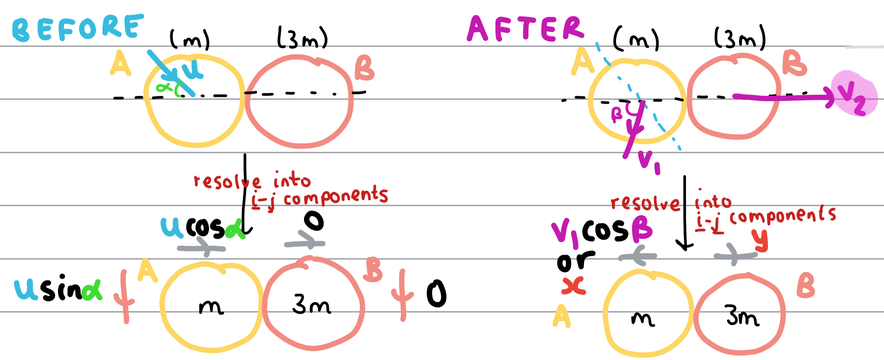
(b) Show that $e > \dfrac{1}{3}$

(4)

(c) Show that $0 < \tan\alpha \leqslant \dfrac{1}{\sqrt{2}}$

(5)

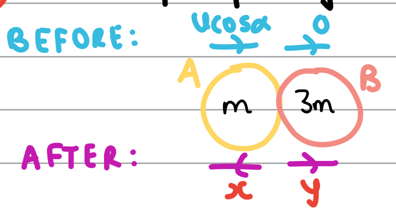
(a) Let's break the diagram in Fig 1 down into a before and after the collision happens



Question 7 continued

... treating as a standard elastic collision in 1D :

$\rightarrow +$ ... first, using PCLM :



formula: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$

subbing into above:

$\cancel{m}(u\cos\alpha) = \cancel{m}(-x) + 3\cancel{m}(y)$

cancel m's and expand brackets

$u\cos\alpha = -x + 3y$

$\Rightarrow -x + 3y = u\cos\alpha$ -①

... next, because both speeds after are unknown, use NEL:

formula: $e = \dfrac{\text{speed of separation}}{\text{speed of approach}} = \dfrac{y - (-x)}{u\cos\alpha}$

expand brackets

$e = \dfrac{y + x}{u\cos\alpha}$

$\times u\cos\alpha$ $\quad$ $u\cos\alpha$

$\Rightarrow y + x = eu\cos\alpha$ ②

the question asks for 'y', or v_1 ∴ elim. 'x'

① + ②

$$\begin{aligned} -x + 3y &= u\cos\alpha \\ + \quad x + y &= eu\cos\alpha \\ \hline 4y &= u\cos\alpha + eu\cos\alpha \end{aligned}$$

factorise $u\cos\alpha$ out :

$4y = u\cos\alpha(1 + e)$

$\div 4$ $\quad$ $\div 4$

$\boxed{y = \frac{u}{4}\cos\alpha(1+e)}$

(b) we have to use the fact that A is rotated 90° as our only source of inequality - hence $x > 0$ (as acknowledged on our diagram)

↳ hence, we want to find x (elim. y):

3×② -①

$$\begin{aligned} 3x + 3y &= 3eu\cos\alpha \\ - \quad -x + 3y &= u\cos\alpha \\ \hline 4x &= 3eu\cos\alpha - u\cos\alpha \end{aligned}$$

factorise $u\cos\alpha$ out

$4x = u\cos\alpha(3e - 1)$

$\div 4$ $\quad$ $\div 4$

$x = \frac{u}{4}\cos\alpha(3e-1)$

sub into our source of inequality

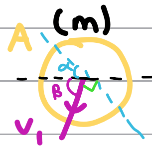
Question 7 continued

$$\frac{u}{4}(3e-1)\cos\alpha > 0$$

$$\Rightarrow 3e-1 > 0$$

$$\Rightarrow \boxed{e > \tfrac{1}{3}}$$

(c) now that the angle α is involved we're prompted to looking at the parallel and perpendicular components for A after the collision



$$\Rightarrow \tan\alpha = \frac{\text{parallel}}{\text{perp.}}$$

$$= \frac{\frac{1}{4}(3e-1)\cancel{u}\cos\alpha}{\cancel{u}\sin\alpha}$$

cancel u's and use $\frac{\sin\alpha}{\cos\alpha}$

$$\tan\alpha = \frac{1}{4}(3e-1)\frac{1}{\tan\alpha}$$

$\times\tan\alpha$ $\times\tan\alpha$

$$\tan^2\alpha = \frac{1}{4}(3e-1)$$

know $0 \le e \le 1$

$\therefore$ as $e \le 1$,

$$\tan^2\alpha \le \frac{1}{4}(2)$$

$$\Rightarrow \tan^2\alpha \le \tfrac{1}{2}$$

but $0 < \alpha \le 90^\circ$ where tan is +ve

$\therefore$ full range $\Rightarrow \boxed{0 \le \tan^2\alpha \le \tfrac{1}{2}}$

Question 7 continued

Question 7 continued

(Total for Question 7 is 15 marks)

TOTAL FOR PAPER IS 75 MARKS